Chapter 3. Interpolation

Section 1. Lagrange Interpolation

In many application problems, we need to use polynomials to approximate some complicated or unknown functions, which usually are given in the form of discrete values, i.e., the function values are known only at given points. Lagrange interpolation determines a polynomial of order n for a given set of $n+1$ points.

1. Linear interpolation. (Burden & Faires, 3.1)

If a set of two points, $(x_0, f(x_0))$, $(x_1, f(x_1))$, are given, we want to find a linear polynomial (degree 1) which passes through these two points. Let

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

and define

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

It is easy to check that $P(x)$ is the required polynomial.

2. General case. (Burden & Faires, 3.1)

Similar to the linear case, we want to find a polynomial of degree n which passes through $n+1$ points

$$(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n)),$$

Let

$$P_n(x) = f(x_0) L_{n,0}(x) + f(x_1) L_{n,1}(x) + \cdots + f(x_n) L_{n,n}(x)$$

where $L_{n,k}(x), k = 0, 1, \cdots, n$, are polynomials of degree n, which are to be determined. Since

$$P_n(x_k) = f(x_k), \quad k = 0, 1, \cdots, n$$

the polynomial $L_{n,k}$ should satisfies the condition

$$L_{n,k}(x_k) = 1, \quad L_{n,k}(x_i) = 0, \quad i \neq k$$

Thus, $L_{n,k}(x)$ contains the factor

$$(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)$$

Use $L_{n,k}(x_k) = 1$ we obtain

$$\frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

Theorem 3.3. (error term of Lagrange interpolation) Suppose that $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x \in [a, b]$, there exists a number $\xi \in (a, b)$ shch that

$$f(x) = P_n(x) + \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \quad \blacksquare$$

Section 2. Hermite interpolation.

In the Lagrange interpolation, at each node, the polynomial has the same value as the function value. However, in some applicaton problems, we may want the approximate polynomial has the same slope as the function. This results in a different kind of polynomial interpolation, the Hermite interpolation.

Theorem 3.9. (Hermite interpolation) If $f \in C^1[a, b]$ and $x_0, x_1, \cdots, x_n \in [a, b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \cdots, x_n$ is the Hermite polynomial of degree at most $2n + 1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

where

$$\begin{aligned} H_{n,j}(x) &= [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x) \\ \hat{H}_{n,j}(x) &= (x - x_j)L^2_{n,j}(x) \end{aligned}$$

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

for some $\xi \in (a, b)$.

Section 3. Piecewise polynomial interpolation.

For some problems, high order interpolation may not give satisfactory results. For example (Runge), let

$$f(x) = \frac{1}{1 + x^2}, \quad -5 \le x \le 5$$

If we take $x_k = -5 + 10k/n, k = 0, 1, \cdots, n$, then the resulting Lagrange interpolation polynomial only converges for $|x| \le 3.63$, as $n \to \infty$. In this case, a piecewise interpolation using low order polynomials would give much better results. The simplest one is the piecewise linear interpolation. Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

we want to find the piecewise linear function of the form

$$I_1(x) = \sum_{k=0}^{n} f(x_k) l_k(x)$$

It is easy to see that $l_k(x)$ should be a piecewise linear function and satisfy the conditions

$$l_k(x_k) = 1, \quad L_k(x_i) = 0, \quad i \neq k$$

Thus, $l_k(x) = 0$ in all subintervals except the two subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, i.e.,

$$l_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}}, & x_{k-1} \leq x \leq x_k \\ \frac{x - x_{k+1}}{x_k - x_{k+1}}, & x_k \leq x \leq x_{k+1} \\ 0, & x \text{ in other subintervals} \end{cases}$$

Similarly, we can construct the piecewise quadratic interpolation $I_2(x)$.

Section 4. Splines

For the piecewise linear or piecewise quadratic interpolations, the function $I_1(x)$ or $I_2(x)$ is continuous in the interval $[a, b]$, but not smooth in general, i.e., $I_1'(x)$ or $I_2'(x)$ is discontinuous at the nodes $x_k, k = 1, 2, \cdots n-1$. If a smooth piecewise polynomial is required, we need the **spline interpolation**. In the spline interpolations, the cubic spline interpolation is the most polular one.

Definition 3.10 (cubic spline interpolation) Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolation S for f is a function that satisfies the following conditions,

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \cdots, n-1$

(b) $S_j(x_j) = f(x_j)$, and $S_j(x_{j+1}) = f(x_{j+1})$, for each $j = 0, 1, \cdots, n-1$

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$, for each $j = 0, 1, \cdots, n-2$

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$, for each $j = 0, 1, \cdots, n-2$

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, for each $j = 0, 1, \cdots, n-2$

(f) One of the following sets of boundary conditions is satisfied

(i) $S''(x_0) = S''(x_n) = 0$ (**free or natural boundary**)

(ii) $S'(x_0) = f(x_0)$ and $S'(x_n) = f(x_n)$ (**clamped boundary**)

Theorem 3.11. (existence and uniqueness of the natural spline) If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \cdots, x_n$, i.e., a spline interpolant that satisfies the boundary condition $S''(a) = S''(b) = 0$.

Theorem 3.12. (existence and uniqueness of the clamped spline) If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$, and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, \cdots, x_n$, i.e., a spline interpolant that satisfies the boundary condition $S'(a) = f'(a)$ and $S'(b) = f'(b)$.

Theorem 3.13. (error of the clamped spline) Let $f \in C^4[a,b]$ with $\max_{a \leq x \leq b} |f^{(4)}(x)| = M$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes $a = x_0 < x_1 < \cdots < x_n = b$, then for all $x \in [a,b]$ we have

$$|f(x) - S(x)| \leq \frac{5M}{384} \max_{0 \leq j \leq n-1} (x_{j+1} - x_j)^4$$