Interpolation & Polynomial Approximation

Cubic Spline Interpolation III

Numerical Analysis (9th Edition)

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Outline

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Introduction to Clamped Splines

Example: Clamped Cubic Spline

- In an earlier ▸ Example we found a natural spline S that passes through the points $(1, 2)$, $(2, 3)$, and $(3, 5)$.
- Construct a clamped spline s through these points that has $s'(1) = 2$ and $s'(3) = 1$.

Introduction to Clamped Splines

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Starting Point

Let

$$s_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$$

be the cubic on $[1, 2]$ and the cubic on $[2, 3]$ be

$$s_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$$

and then use the various conditions to determine the 8 constants.

Introduction to Clamped Splines

$$
\begin{aligned}
s_0(x) &= a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\
s_1(x) &= a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3
\end{aligned}
$$

Solution (1/2)

Assemble the various conditions:

Introduction to Clamped Splines

$$\begin{aligned} s_0(x) &= a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\ s_1(x) &= a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \end{aligned}$$

Solution (1/2)

Assemble the various conditions:

$$2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0$$

Introduction to Clamped Splines

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Solution (1/2)

Assemble the various conditions:

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\begin{aligned}
2 = f(1) = a_0, &\quad 3 = f(2) = a_0 + b_0 + c_0 + d_0 \\
3 = f(2) = a_1, &\quad 5 = f(3) = a_1 + b_1 + c_1 + d_1
\end{aligned}
$$

Introduction to Clamped Splines

$$
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s_0(x) &= a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\
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Solution (1/2)

Assemble the various conditions:

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&2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0 \\
&3 = f(2) = a_1, \quad 5 = f(3) = a_1 + b_1 + c_1 + d_1 \\
&s_0'(2) = s_1'(2) \quad \Rightarrow \quad b_0 + 2c_0 + 3d_0 = b_1
\end{aligned}
$$

Introduction to Clamped Splines

$$\begin{aligned} s_0(x) &= a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\ s_1(x) &= a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \end{aligned}$$

Solution (1/2)

Assemble the various conditions:

$$\begin{aligned} 2 = f(1) = a_0, &\quad 3 = f(2) = a_0 + b_0 + c_0 + d_0 \\ 3 = f(2) = a_1, &\quad 5 = f(3) = a_1 + b_1 + c_1 + d_1 \\ s_0'(2) = s_1'(2) &\quad \Rightarrow \quad b_0 + 2c_0 + 3d_0 = b_1 \\ s_0''(2) = s_1''(2) &\quad \Rightarrow \quad 2c_0 + 6d_0 = 2c_1 \end{aligned}$$

Introduction to Clamped Splines

Solution (2/2)

However, the boundary conditions are now

$$s_0'(1) = 2 \quad \Rightarrow \quad b_0 = 2 \quad \text{and} \quad s_1'(3) = 1 \quad \Rightarrow \quad b_1 + 2c_1 + 3d_1 = 1$$

Introduction to Clamped Splines

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Solving this system of equations gives the spline as

$$s(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3, & \text{for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3, & \text{for } x \in [2,3] \end{cases}$$

Outline

Unique Clamped Spline Interpolant

Theorem

If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the clamped boundary conditions $S'(a) = f'(a)$ and $S'(b) = f'(b)$.

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Note: With $S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$, for each $j = 0, 1, \ldots, n - 1$, the proof will rely on two equations established earlier, namely:

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$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

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$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

Unique Clamped Spline Interpolant

$$(A) \qquad b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

Proof (1/5)

Unique Clamped Spline Interpolant

$$(A) \qquad b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

Proof (1/5)

Since $f'(a) = S'(a) = S'(x_0) = b_0$, Equation (A) with $j = 0$ implies

$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

Unique Clamped Spline Interpolant

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$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

Consequently,

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

Unique Clamped Spline Interpolant

$$(A) \qquad b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

Proof (2/5)

Unique Clamped Spline Interpolant

$$(A) \qquad b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

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Similarly,

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

Unique Clamped Spline Interpolant

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Proof (2/5)

Similarly,

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

so equation (A) with $j = n - 1$ implies that

$$\begin{aligned} f'(b) &= \frac{a_n - a_{n-1}}{h_{n-1}} - \frac{h_{n-1}}{3}(2c_{n-1} + c_n) + h_{n-1}(c_{n-1} + c_n) \\ &= \frac{a_n - a_{n-1}}{h_{n-1}} + \frac{h_{n-1}}{3}(c_{n-1} + 2c_n) \end{aligned}$$

Unique Clamped Spline Interpolant

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and $$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

Unique Clamped Spline Interpolant

Proof (3/5)

The equations

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

Unique Clamped Spline Interpolant

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together with

$$2h_0 c_0 + h_0 c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

Unique Clamped Spline Interpolant

Proof (3/5)

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and

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

Unique Clamped Spline Interpolant

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together with

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

and

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

determine the linear system $A\mathbf{x} = \mathbf{b}$:

Unique Clamped Spline Interpolant

Proof (4/5)

where

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

Unique Clamped Spline Interpolant

Proof (5/5)

and

$$\mathbf{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix} \quad \text{with} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Matrix A is strictly diagonally dominant and a linear system with a matrix of this form can be shown ▸ Theorem to have a unique solution for $c_0, c_1, \ldots, c_n$.

Outline

Clamped Cubic Spline Algorithm

Context

To construct the cubic spline interpolant S for the function f defined at the numbers $x_0 < x_1 < \cdots < x_n$, satisfying

$$S'(x_0) = f'(x_0) \qquad \text{and} \qquad S'(x_n) = f'(x_n)$$

where

$$S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

for $x_j \leq x \leq x_{j+1}$.

Clamped Cubic Spline Algorithm

INPUT n; $x_0, x_1, \ldots, x_n$; $a_0 = f(x_0)$, $a_1 = f(x_1), \ldots, a_n = f(x_n)$; $FPO = f'(x_0)$; $FPN = f'(x_n)$.

Clamped Cubic Spline Algorithm

INPUT n; $x_0, x_1, \ldots, x_n$; $a_0 = f(x_0)$, $a_1 = f(x_1), \ldots, a_n = f(x_n)$; $FPO = f'(x_0)$; $FPN = f'(x_n)$.

OUTPUT a_j, b_j, c_j, d_j for $j = 0, 1, \ldots, n-1$

Clamped Cubic Spline Algorithm

INPUT n; $x_0, x_1, \ldots, x_n$; $a_0 = f(x_0)$, $a_1 = f(x_1), \ldots, a_n = f(x_n)$; $FPO = f'(x_0)$; $FPN = f'(x_n)$.

OUTPUT a_j, b_j, c_j, d_j for $j = 0, 1, \ldots, n-1$

Step 1 For $i = 0, 1, \ldots, n-1$ set $h_i = x_{i+1} - x_i$

Clamped Cubic Spline Algorithm

INPUT n; $x_0, x_1, \ldots, x_n$; $a_0 = f(x_0)$, $a_1 = f(x_1), \ldots, a_n = f(x_n)$; $FPO = f'(x_0)$; $FPN = f'(x_n)$.

OUTPUT a_j, b_j, c_j, d_j for $j = 0, 1, \ldots, n-1$

Step 1 For $i = 0, 1, \ldots, n-1$ set $h_i = x_{i+1} - x_i$

Step 2 Set $\alpha_0 = 3(a_1 - a_0)/h_0 - 3FPO$

$$\alpha_n = 3FPN - 3(a_n - a_{n-1})/h_{n-1}$$

Clamped Cubic Spline Algorithm

INPUT $n;\ x_0, x_1, \ldots, x_n;\ a_0 = f(x_0),\ a_1 = f(x_1), \ldots, a_n = f(x_n);$ $FPO = f'(x_0);\ FPN = f'(x_n).$

OUTPUT a_j, b_j, c_j, d_j for $j = 0, 1, \ldots, n-1$

Step 1 For $i = 0, 1, \ldots, n-1$ set $h_i = x_{i+1} - x_i$

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$$\alpha_n = 3FPN - 3(a_n - a_{n-1})/h_{n-1}$$

Step 3 For $i = 1, 2, \ldots, n-1$

$$\text{set } \alpha_i = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1})$$

(Note: In what follows, Steps 4, 5, 6 and part of Step 7 solve a tridiagonal linear system using a Crout Factorization algorithm.)

Natural Cubic Spline Algorithm (Cont'd)

Step 4 Set $l_0 = 2h_0$

$$\mu_0 = 0.5$$

$$z_0 = \alpha_0 / l_0$$

Natural Cubic Spline Algorithm (Cont'd)

Step 4 Set $l_0 = 2h_0$

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Step 5 For $i = 1, 2, \ldots, n-1$

set $l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}$

$$\mu_i = h_i / l_i$$

$$z_i = (\alpha_i - h_{i-1} z_{i-1}) / l_i$$

Natural Cubic Spline Algorithm (Cont'd)

Step 4 Set $l_0 = 2h_0$

$$\mu_0 = 0.5$$

$$z_0 = \alpha_0 / l_0$$

Step 5 For $i = 1, 2, \ldots, n-1$

set $l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}$

$$\mu_i = h_i / l_i$$

$$z_i = (\alpha_i - h_{i-1} z_{i-1}) / l_i$$

Step 6 Set $l_n = h_{n-1}(2 - \mu_{n-1})$

$$z_n = (\alpha_n - h_{n-1} z_{n-1}) / l_n$$

$$c_n = z_n$$

Natural Cubic Spline Algorithm (Cont'd)

Step 4 Set $l_0 = 2h_0$

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Step 5 For $i = 1, 2, \ldots, n-1$

set $l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}$

$$\mu_i = h_i / l_i$$

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Step 6 Set $l_n = h_{n-1}(2 - \mu_{n-1})$

$$z_n = (\alpha_n - h_{n-1} z_{n-1}) / l_n$$

$$c_n = z_n$$

Step 7 For $j = n-1, n-2, \ldots, 0$

set $c_j = z_j - \mu_j c_{j+1}$

$$b_j = (a_{j+1} - a_j)/h_j - h_j(c_{j+1} + 2c_j)/3$$

$$d_j = (c_{j+1} - c_j)/(3h_j)$$

Natural Cubic Spline Algorithm (Cont'd)

Step 4 Set $l_0 = 2h_0$

$$\mu_0 = 0.5$$

$$z_0 = \alpha_0 / l_0$$

Step 5 For $i = 1, 2, \ldots, n-1$

set $l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}$

$$\mu_i = h_i / l_i$$

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Step 8 OUTPUT $(a_j, b_j, c_j, d_j$ for $j = 0, 1, \ldots, n-1)$ & STOP

Outline

Clamped Cubic Spline for $f(x) = e^x$

Example: 3 Data Points

A previous ▸ Example used a natural spline and the data points

$$(0, 1), \quad (1, e), \quad (2, e^2) \text{ and } (3, e^3)$$

to form a new approximating function $S(x)$. Determine the clamped spline $s(x)$ that uses this data and the additional information that, since $f'(x) = e^x$, so $f'(0) = 1$ and $f'(3) = e^3$.

Clamped Cubic Spline for $f(x) = e^x$

Solution (1/5)

As before, we have $n = 3$, $h_0 = h_1 = h_2 = 1$, $a_0 = 0$, $a_1 = e$, $a_2 = e^2$, and $a_3 = e^3$. This together with the information that $f'(0) = 1$ and $f'(3) = e^3$ gives the the matrix A and the vectors **b** and **x** with the forms Original Form:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3(e-2) \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 3e^2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Clamped Cubic Spline for $f(x) = e^x$

Solution (2/5)

The vector-matrix equation $A\mathbf{x} = \mathbf{b}$ is equivalent to the system of equations:

$$
\begin{aligned}
2c_0 + c_1 &= 3(e-2), \\
c_0 + 4c_1 + c_2 &= 3(e^2 - 2e + 1), \\
c_1 + 4c_2 + c_3 &= 3(e^3 - 2e^2 + e), \\
c_2 + 2c_3 &= 3e^2.
\end{aligned}
$$

Clamped Cubic Spline for $f(x) = e^x$

Solution (3/5)

Solving this system simultaneously for c_0, c_1, c_2 and c_3 gives, to 5 decimal places,

$$c_0 = \frac{1}{15}(2e^3 - 12e^2 + 42e - 59) = 0.44468,$$
$$c_1 = \frac{1}{15}(-4e^3 + 24e^2 - 39e + 28) = 1.26548,$$
$$c_2 = \frac{1}{15}(14e^3 - 39e^2 + 24e - 8) = 3.35087,$$
$$c_3 = \frac{1}{15}(-7e^3 + 42e^2 - 12e + 4) = 9.40815.$$

Clamped Cubic Spline for $f(x) = e^x$

Step 7 For $j = n-1, n-2, \ldots, 0$

$$\text{set } c_j = z_j - \mu_j c_{j+1}$$
$$b_j = (a_{j+1} - a_j)/h_j - h_j(c_{j+1} + 2c_j)/3$$
$$d_j = (c_{j+1} - c_j)/(3h_j)$$

Solution (4/5)

Solving for the remaining constants (using Step 7 of the Algorithm)

Clamped Cubic Spline for $f(x) = e^x$

Step 7 For $j = n-1, n-2, \ldots, 0$

$$\begin{aligned} \text{set } c_j &= z_j - \mu_j c_{j+1} \\ b_j &= (a_{j+1} - a_j)/h_j - h_j(c_{j+1} + 2c_j)/3 \\ d_j &= (c_{j+1} - c_j)/(3h_j) \end{aligned}$$

Solution (4/5)

Solving for the remaining constants (using Step 7 of the Algorithm) gives

$$b_0 = 1.00000, \quad b_1 = 2.71016, \quad b_2 = 7.32652$$

Clamped Cubic Spline for $f(x) = e^x$

Step 7 For $j = n-1, n-2, \ldots, 0$

$$\text{set } c_j = z_j - \mu_j c_{j+1}$$
$$b_j = (a_{j+1} - a_j)/h_j - h_j(c_{j+1} + 2c_j)/3$$
$$d_j = (c_{j+1} - c_j)/(3h_j)$$

Solution (4/5)

Solving for the remaining constants (using Step 7 of the Algorithm) gives

$$b_0 = 1.00000, \quad b_1 = 2.71016, \quad b_2 = 7.32652$$

and

$$d_0 = 0.27360, \quad d_1 = 0.69513, \quad d_2 = 2.01909$$

Clamped Cubic Spline for $f(x) = e^x$

Solution (5/5)

This gives the clamped cubic spine

$$s(x) = \begin{cases} 1 + x + 0.44468x^2 + 0.27360x^3 & \text{if } 0 \leq x < 1 \\ 2.71828 + 2.71016(x-1) + 1.26548(x-1)^2 + 0.69513(x-1)^3 & \text{if } 1 \leq x < 2 \\ 7.38906 + 7.32652(x-2) + 3.35087(x-2)^2 + 2.01909(x-2)^3 & \text{if } 2 \leq x \leq 3 \end{cases}$$

The graph of the clamped spline and $f(x) = e^x$ are so similar that no difference can be seen.

Outline

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

Example: The Integral of a Spline

Approximate the integral of $f(x) = e^x$ on $[0, 3]$, which has the value

$$\int_0^3 e^x\,dx = e^3 - 1 \approx 20.08553692 - 1 = 19.08553692,$$

by piecewise integrating the clamped cubic spline that approximates f on this integral.

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

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Note: From the previous example, the clamped cubic spine $s(x)$ that approximates $f(x) = e^x$ on $[0,3]$ is described piecewise by

$$s(x) = \begin{cases} 1 + x + 0.44468x^2 + 0.27360x^3 & \text{if } 0 \le x < 1 \\ 2.71828 + 2.71016(x-1) + 1.26548(x-1)^2 + 0.69513(x-1)^3 & \text{if } 1 \le x < 2 \\ 7.38906 + 7.32652(x-2) + 3.35087(x-2)^2 + 2.01909(x-2)^3 & \text{if } 2 \le x \le 3 \end{cases}$$

Clamped Cubic Spline for $\int_0^3 e^x \, dx$

Solution (1/3)

Because the data is equally spaced, piecewise integrating the clamped spline results in the same formula as that for the naturual cubic spline,

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

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Because the data is equally spaced, piecewise integrating the clamped spline results in the same formula as that for the naturual cubic spline, that is:

$$\begin{aligned}\int_0^3 s(x)\,dx &= (a_0 + a_1 + a_2) + \frac{1}{2}(b_0 + b_1 + b_2) \\ &\quad + \frac{1}{3}(c_0 + c_1 + c_2) + \frac{1}{4}(d_0 + d_1 + d_2)\end{aligned}$$

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

Solution (2/3)

Hence the integral approximation is

$$\begin{aligned}\int_0^3 s(x)\,dx &= (1 + 2.71828 + 7.38906) + \frac{1}{2}(1 + 2.71016 + 7.32652)\\ &\quad + \frac{1}{3}(0.44468 + 1.26548 + 3.35087)\\ &\quad + \frac{1}{4}(0.27360 + 0.69513 + 2.01909)\\ &= 19.05965.\end{aligned}$$

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

Solution (3/3)

The absolute error in the integral approximation using the clamped and natural splines are

$$\begin{aligned} \text{Natural:} &\quad |19.08554 - 19.55229| = 0.46675 \\ \text{Clamped:} &\quad |19.08554 - 19.05965| = 0.02589 \end{aligned}$$

Clamped Cubic Spline for $\int_0^3 e^x \, dx$

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For integration purposes, the clamped spline is vastly superior.

Clamped Cubic Spline for $\int_0^3 e^x\,dx$

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For integration purposes, the clamped spline is vastly superior. This should be no surprise since the boundary conditions for the clamped spline are exact, whereas for the natural spline we are essentially assuming that, since $f''(x) = e^x$,

$$0 = S''(x) \approx f''(0) = e^1 = 1 \quad \text{and} \quad 0 = S''(3) \approx f''(3) = e^3 \approx 20$$

Clamped Cubic Spline: Accuracy Statements

To conclude this section, we list an error-bound formula for the cubic spline with clamped boundary conditions. The proof of this result can be found in [Schul], pp. 57–58.

Clamped Cubic Spline: Accuracy Statements

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Theorem

Let $f \in C^4[a, b]$ with $\max_{a \leq x \leq b} |f^{(4)}(x)| = M$. If S is the unique clamped cubic spline interpolant to f with respect to the nodes $a = x_0 < x_1 < \cdots < x_n = b$, then for all x in $[a, b]$,

$$|f(x) - S(x)| \leq \frac{5M}{384} \max_{0 \leq j \leq n-1} (x_{j+1} - x_j)^4$$

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A fourth-order error-bound result also holds in the case of natural boundary conditions, but it is more difficult to express.

Clamped Cubic Spline: Accuracy Statements

Natural Boundary Conditions: Final Remark

The natural boundary conditions will generally give less accurate results than the clamped conditions near the ends of the interval $[x_0, x_n]$ unless the function f happens to nearly satisfy

$$f''(x_0) = f''(x_n) = 0$$

Questions?

Reference Material

Natural Cubic Spline

Example: 3 Data Values

Construct a natural cubic spline that passes through the points $(1,2)$, $(2,3)$, and $(3,5)$.

The Resulting Spline

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & \text{for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, & \text{for } x \in [2,3] \end{cases}$$

Return to Clamped Spline Example

Strictly Diagonally Dominant Matrices

Theorem

- A strictly diagonally dominant matrix A is nonsingular.
- Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off errors.

◂ Return to Clamped Spline Uniqueness Proof

Clamped Spline Interpolant: Linear System $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix} \quad \text{with} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Return to Clamped Spline Example ($f(x) = e^x$)

Natural Spline Interpolant

Example: $f(x) = e^x$

Use the data points $(0, 1)$, $(1, e)$, $(2, e^2)$, and $(3, e^3)$ to form a natural spline $S(x)$ that approximates $f(x) = e^x$.

Answer

The natural cubic spine is described piecewise by

$$S(x) = \begin{cases} 1 + 1.46600x + 0.25228x^3 & \text{for } x \in [0, 1] \\ 2.71828 + 2.22285(x-1) + 0.75685(x-1)^2 + 1.69107(x-1)^3 & \text{for } x \in [1, 2] \\ 7.38906 + 8.80977(x-2) + 5.83007(x-2)^2 - 1.94336(x-2)^3 & \text{for } x \in [2, 3] \end{cases}$$

Return to Clamped Spline Example ($f(x) = e^x$)